Course Website: http://www.uccs.edu/rcascara/1Math340

eCollege/eCompanion: http://www.uccs.edu/online/loci.htm

Introduction: Differential Equations

\[ t = \text{independent variable} \]
\[ x = \text{dependent variable} \]
\[ X = x(t) \]

Often times, \( x = x(t) \) is given by a rule

\[ \frac{dx}{dt} = \text{rate of change of } x \text{ w.r.t. } t \]

is given as a function of \( t \) and/or \( x \)

\[ \frac{dx}{dt} = f(t, x) \]

\( 1^{st} \text{ order differential eq.} \)

\[ \frac{dx}{dt}, \ x(t), \ x \]
Ex: Exponential growth:

\[ x(t) = \text{population at time } t \]

\[ \frac{dx}{dt} = ax, \quad a = \text{constant } (a > 0) \]

is a 1st order differential eq:

\[ \frac{dx}{dt} = f(x) \]

Q: What are the solutions of \( \frac{dx}{dt} = ax \)?

\[ x(t) = e^{at} \]

\[ x'(t) = ae^{at} \]

\[ = a \cdot x(t) \]

One can check

\[ x(t) = 3e^{at} \quad \text{also solves } \quad x' = ax \]

\[ x(t) = ce^{at}, \quad c = \text{arbitrary constant} \]
If the original problem indicates an "initial condition" 
\[ X(0) = 1247 \]
then we are interested in finding THE solution which satsfies this property
\[ X(t) = C e^{at} \]
At \( t = 0 \) : 
\[ X(0) = C e^{a \cdot 0} = C \cdot 1 = C \]
\[ 1247 = C \]

\[ \Rightarrow X(t) = 1247 e^{at} \] is the solution of the initial value problem \( \frac{dx}{dt} = ax \), \( X(0) = 1247 \)

---

Easiest differential eq to solve:

\[ \frac{dx}{dt} = f(t) \Rightarrow x(t) = \int f(t) \, dt \]

\( \text{Ex: Solve } \frac{dx}{dt} = 3t^2 \Rightarrow x(t) = \int 3t^2 \, dt \)

\[ x(t) = t^3 + C \]
\[ \text{Ex: Solve } \frac{dx}{dt} = e^{-t^2} \Rightarrow x(t) = \int e^{-t^2} \, dt \]

cannot be explicitly written in terms of elementary functions; polynomials \( t, t^2, 3t^5 + 2t + 4 \),

rational functions \( \frac{t + 2}{t^2 + 4} \),

exp, log,

trig functions \( \sin, \cos, \tan \),

inverse trig \( \sin^{-1}, \cos^{-1}, \tan^{-1} \).

Given an initial condition

\[ x(1) = 3 \]

Solve \( \frac{dx}{dt} = e^{-t^2} \). (use definite integrals)

\[ x(t) = \int_1^t e^{-s^2} \, ds + C \]

When \( t = 1 \Rightarrow 3 = x(1) = \int_1^1 e^{-s^2} \, ds + C \)

\[ 3 = C \]

\[ t \quad x(t) = 3 + \int_1^t e^{-s^2} \, ds. \]
Applications:

\[ x(t) = \text{position at time } t \]

\#25 (page 12) \[ v(t) = \frac{dx}{dt}(t) = \text{velocity (speed) at time } t \]

\[ a(t) = \frac{d^2x}{dt^2}(t) = \text{acceleration at time } t \]

**Given**

\[ v(0) = 60 \text{ (km/h)}, \quad x(0) = 0 \]

\[ a(t) = -2500 \text{ (km/h^2)} \text{ for all } t \]

**determine the stopping distance } x(t^*) \text{ when } v(t^*) = 0.\]

\[ a(t) = \frac{d^2x}{dt^2} = -2500 \implies v(t) = \frac{dx}{dt} = \int -2500 \, dt \]

\[ \frac{dx}{dt} = -2500t + c_1 \]

\[ x(t) = \int (-2500t + c_1) \, dt \]

\[ = -1250t^2 + c_1 t + c_2 \]

\[ c_1, c_2 = ? \]
Match Initial conditions

\[ v(0) = 60 \implies 60 = -2500 \cdot 0 + c_1 \]
\[ \implies c_1 = 60 \]

\[ x(0) = 0 \implies 0 = 0 + 0 + c_2 \implies c_2 = 0 \]

\[ \implies x(t) = -1250 \cdot t^2 + 60 \cdot t \]

To find the stopping time \( t = t^* \)

\[ 0 = v(t) = -2500 \cdot t + 60 \]

\[ 2500 \cdot t = 60 \]

\[ t^* = \frac{60}{2500} = 0.024 \text{ (h)} \]

Stopping distance

\[ x(t^*) = -1250(t^*)^2 + 60 \cdot t^* \]

\[ = 0.72 \text{ (km)} \]

First truly relevant class of problems.
Separable equations (1st order)

\[ \frac{dx}{dt} = \frac{x^2 \cdot \cos t}{\text{function of } x \text{ only}} \]

Separate the variables:

\[ \int \frac{dx}{x^2} = \int \cos t \, dt \]

\[ -\frac{1}{x} = \sin t + C \]

\[ \frac{1}{x} = - (\sin t + C) \]

\[ x = \frac{1}{-(\sin t + C)} = - \frac{1}{\sin t + C} \]

\[ x(t) = \frac{1}{C - \sin t} \]