Lecture 2  Heat Equation (cont.)

One-dimensional:

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

\( u = u(x,t) \)

\( u \) - temperature in an 1-D rod

What happens if the medium is finite \( x \in [0, L] \)?

Boundary conditions: At each end point \( x = 0 \) and \( x = L \), we need to specify the physics.

At left-end point \( (x=0) \)

(i) \( u(0,t) = \alpha \) (constant in \( t \))

(prescribed temperature) \( \text{Dirichlet boundary condition} \)

(ii) \( \frac{\partial u}{\partial x}(0,t) = \beta \) (constant in \( t \))

(prescribed heat flux) \( \text{Neumann boundary condition} \)
\[ \phi = -k_0 \frac{du}{dx} \quad \text{(Fourier Law)} \]

\( \tag{11} \frac{du}{dx} \mid_{x=0} + \sum u(x) = \delta \quad \text{Robin boundary condition} \)

(Newton's law of cooling)

\[ \frac{du}{dx} = c(u_o - u) \]

Equilibrium states \[ u^*(x) \quad \text{(independent on} + \text{)} \]

If we require \[ u^*(x) \] to be a solution of the heat equation + boundary conditions

\[ \Rightarrow \quad 0 = \frac{\partial u^*}{\partial t} = k \frac{\partial^2 u^*}{\partial x^2} \]

+ boundary conditions

\[ \Rightarrow \quad \frac{d^2 u^*}{dx^2} = 0 \quad \Rightarrow \quad u^* = \text{linear in} x \]

\[ u^*(x) = Cx + D \]

\( \Sigma x: 1.4.1 \) (d)

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

\[ u(0,t) = T, \quad \frac{\partial u}{\partial x} (L,t) = \alpha \]
Equilibrium: \( 0 = \frac{\partial^2 u}{\partial x^2} \Rightarrow u = \text{linear} \times x \)

\[ u(x) = Cx + D \]

At \( x = 0 \Rightarrow T = C_0 + D \Rightarrow \boxed{D = T} \)

\( x = L \Rightarrow \alpha = C \Rightarrow \boxed{C = \alpha} \)

\[ \Rightarrow \boxed{u^*(x) = \alpha x + T} \]

1.4.4: Insulated rod ends

no flux across \( x = 0 \) and \( x = L \)

\[ \frac{\partial u}{\partial x} (0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x} (L,t) = 0 \]

Under these conditions, the total thermal energy is constant.

\[ e(x,t) = \text{energy density (per volume)} \]
Total energy = \( \iiint_V e(x,t) \, dV \)

= \( \int_0^L e(x,t) \, A \, dx \)

= \( \int_0^L \varepsilon \sigma \ u(x,t) \, A \, dx \)

Total energy = \( K \int_0^L u(x,t) \, dx \)

at time \( t \)

\[ E(t) = \int_0^L u(x,t) \, dx \]

Show \( \frac{d}{dt} E(t) = 0 \) !!!!

\[ \frac{d}{dt} E(t) = \frac{d}{dt} \int_0^L u(x,t) \, dx \]

= \( \int_0^L \frac{\partial u}{\partial t} (x,t) \, dx \)

= \( \int_0^L k \frac{\partial^2 u}{\partial x^2} (x,t) \, dx \)

(Find, Then, of Calc)

= \( k \left( \frac{\partial u}{\partial x} \right) \bigg|_{x=L} - \left( \frac{\partial u}{\partial x} \right) \bigg|_{x=0} \)
\[ E(t) = E(0) + kt(\phi_L - \phi_0) \]

(Linear in \( t \))

Homework (b)
\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + Q \]

external source.

\[ \alpha = \text{constant in } t. \]

1.4.11 \[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x \]

\[ u(x,0) = f(x), \quad \frac{\partial u}{\partial x} (0,t) = \beta, \frac{\partial u}{\partial x} (L,t) \]

Note: Equilibrium exists for problems with time-independent boundary and extend source.

1.5 Heat equation in 2D and 3D

3D \[ u = u(x, y, z, t) \]

Temperature at location \((x, y, z)\) at time \(t\)
\[ \frac{\partial u}{\partial t} = k \Delta u \]

where \( \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \) \quad \text{Laplacian}

\[
\text{(in 2D: } u = u(x,y,t) , \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})\]

\( \text{it involves the divergence theorem in 3D} \)
\( \text{(see Calc III)} \)

Boundary conditions?

\( \text{(I) } u = \text{const on } \partial \Omega \)

\( \text{(II) } \frac{\partial u}{\partial n} = \text{const on } \partial \Omega \)

where \( \vec{n} \) = outward normal

\[ \frac{\partial u}{\partial n} = \vec{n} \cdot \nabla u \]

\( \text{(III) } \frac{\partial u}{\partial n} + \gamma u = \delta \text{ on } \partial \Omega \)