Methods in Applied Math—Fall 2008

LECTURE 26  Green's Functions

Solve \( u'' = f \) on \([0, L]\)

with B.C. \( u(0) = 0, u(L) = 0 \)

\[
\left( \begin{array}{c}
L(u) = u'' , \quad u(0) = 0 , \quad u(L) = 0 \\
\end{array} \right)
\]

\( L(u) = f \Rightarrow u = L^{-1}(f) \)

Remark: \( u(x) \sim u_j \quad j = 0, N \)

\( u''(x) \approx \frac{u_{j+1} + u_{j-1} - 2u_j}{(\Delta x)^2} \)

\[
\begin{cases}
\frac{u''(x_j)}{\Delta x^2} = f(x_j) \\
u(0) = 0, u(L) = 0
\end{cases}
\]

\[
\begin{cases}
u_{j+1} + u_{j-1} - 2u_j \\
\Delta x^2 = f_j
\end{cases}
\]

\[
u_0 = 0, \quad u_N = 0
\]

How to solve the continuous problem??
\[
\begin{align*}
\begin{cases}
  u''(x) = f(x) \\
  u(0) = u(L) = 0
\end{cases} 
\Rightarrow u(x) &= \int_0^L G(x, y) f(y) \, dy \\
\text{for some } G = G(x, y) \\
\text{Green's function}
\end{align*}
\]

\[
\begin{align*}
  u''(x) &= f(x) \\
  \Rightarrow u'(x) &= u'(0) + \int_0^x f(y) \, dy \\
  \Rightarrow u(x) &= u(0) + u'(0) x + \int_0^x \int_0^y f(z) \, dz \, dy
\end{align*}
\]

\[
\begin{align*}
  u(0) &= 0 \\
  0 &= u(L) = u'(0) L + \int_0^L \left( \int_0^y f(z) \, dz \right) \, dy \\
  \Rightarrow u'(0) &= -\frac{1}{L} \int_0^L \left( \int_0^y f(z) \, dz \right) \, dy \\
  \Rightarrow u(x) &= -\frac{x}{L} \int_0^L \left( \int_0^y f(z) \, dz \right) \, dy + \int_0^x \int_0^y f(z) \, dz \, dy \\
  &= \ldots \\
  &= \int_0^L G(x, y) f(y) \, dy
\end{align*}
\]
\[
\int_0^L \left( \int_0^L f(z) \, dz \right) \, dy = \int_0^L \int_0^y f(z) \, dz \, dy = \int_0^L \int_0^{L-y} f(z) \, dz \, dy
\]

In the end,
\[
G(x,y) = \begin{cases} 
  -\frac{x}{L} (L-y), & x < y \\
  -\frac{y}{L} (L-x), & x > y 
\end{cases}
\]

\[
G(x,y) = G(y,x)
\]

The key property of \(G(x,y)\) is the following:
\[
G(x, y) \text{ is a solution of the diff. eq.} \quad L(u)_{xy} = \delta(x-y)
\]
\[ \delta = \text{Dirac delta function} \]

\[ \delta(x) = \begin{cases} 
0, & x \neq 0 \\
\infty, & x = 0 
\end{cases} \]

\[ \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \]

The "derivative" of
\[ H(x) = \begin{cases} 
0, & x < 0 \\
1, & x > 0 
\end{cases} \]

is \( \delta = \delta(x) \).
\[ L[G] = \delta(x-y) \]

\[ G_y(x) = G(x,y) \]

Why? \[
\int_{-\infty}^{\infty} L[G] f(y) \, dy = \int_{-\infty}^{\infty} \delta(x-y) f(y) \, dy
\]

\[
\int_{-\infty}^{\infty} G_x''(y) f(y) \, dy = f(x)
\]

\[
\left( \int_{-\infty}^{\infty} G_x(y) f(y) \, dy \right)'' = f(x)
\]

\[ u'' = f \]