Section 7.7 - Differential Equations

A differential equation is an equation that contains a derivative. It relates an unknown with its rate of change.

\[ y = xy' \quad (1^{\text{st}} \text{ order}) \]

\[ \frac{d^2 y}{dx^2} x + \frac{dy}{dx} x^2 = y \quad (2^{\text{nd}} \text{ order}) \]

The order of a differential equation is the order of the highest derivative that occurs in the equation.

The solution of a differential equation is an equation.

\[ f'(x) = f(x) \]

The solution is \( f(x) = Ce^x \).

Determine whether the following are solutions of \( y'' + 2y' + y = 0 \).

a) \( y = e^t \)

Take the derivatives, plug them in, see if it works.

\[ y' = e^t \]
\[ y'' = e^t \]
\[ e^t + 2e^t + e^t \neq 0 \quad \text{No.} \]
b) \(y = t e^{-t}\)
\[y' = t(-e^{-t}) + e^{-t} = e^{-t}(1-t)\]
\[y'' = e^{-t}(-1) + (1-t)(-e^{-t})\]
\[= -e^{-t} - e^{-t} + te^{-t} = te^{-t} - 2e^{-t}\]
\[y'' + 2y' + y = 0\]
\[te^{-t} - 2e^{-t} + 2e^{-t}(1-t) + te^{-t} \neq 0\]
\[2te^{-t} - 2e^{-t} + 2e^{-t} - 2te^{-t} \neq 0\]

Yes

\[\text{Solve } \frac{dy}{dx} = x^2 + 3x\]
\[dy = (x^2 + 3x) \, dx\]
\[y = \frac{1}{2}x^3 + \frac{3}{2}x^2 + c\]

**Separable equations**

A first order differential equation where \(\frac{dy}{dx}\) can be written as \(g(x) \cdot f(y)\).

\[\frac{dy}{dx} = g(x) \cdot f(y)\]
\[\frac{dy}{f(y)} = g(x) \, dx\]
\[\int \frac{dy}{f(y)} = \int g(x) \, dx\]

This will give defined implicitly in terms of \(x\).
If it's feasible, will solve for \(y\) explicitly.
Solve \( y' = xy \)

Rewrite as \( \frac{dy}{dx} = xy \)

We assume \( y \neq 0 \)

\[
\int \frac{dy}{y} = \int x \, dx \quad \text{Put all } y \text{'s on one side, } x \text{'s on other.}
\]

This comes from the case where \( y = 0 \).

\[
\ln|y| = \frac{x^2}{2} + C
\]

\[
|y| = e^{\frac{x^2}{2}} + C
\]

\[
y = \pm e^{\frac{x^2}{2}}
\]

\[
y = Ke^{\frac{x^2}{2}} \quad \text{where } K = \pm e^{C}, \quad 0
\]

p.419 #11 \( \frac{du}{dt} = 2t + \sec^2 t \)

with \( u(0) = -5 \)

\[
\int 2u \, du = \int 2t + \sec^2 t \, dt
\]

\[
u^2 = t^2 + \tan t + C
\]

\[
u = \pm \sqrt{t^2 + \tan t + C}
\]

\[
u(0) = \pm \sqrt{0^2 + \tan 0 + C} \quad \mp 5
\]
\[- \sqrt{c} = -5 \]
\[ \sqrt{c} = 5 \]
\[ c = 25 \]
\[ n(t) = - \sqrt{t^2 + \tan t} + 25 \]

(46) 1000 L water
0.05 kg salt/L @ rate 5L/min
0.04 kg salt/L @ rate 10L/min
Solution drains @ 15L/min

How much salt after t minutes?

Let \( x(t) = \) amount of salt at time \( t \)

\[
\frac{dx}{dt} = \left( \frac{0.05 \text{ kg}}{L} \right) \left( \frac{5 \text{ L}}{\text{min}} \right) + \left( \frac{0.04 \text{ kg}}{L} \right) \left( \frac{10 \text{ L}}{\text{min}} \right) - \frac{x(t) \text{ kg} \cdot 15 \text{ L}}{1000 \text{ L} \cdot \text{min}}
\]

rate of change of amount of salt

\[
\frac{dx}{dt} = 0.25 + 0.4 - 0.015x
\]

\[
= 0.65 - 0.015x
\]

\[
\frac{dx}{dt} = \frac{650 - 15x}{1000}
\]

\[
\frac{dy}{dt} = \frac{130 - 3x}{200}
\]

\[
\frac{dx}{130 - 3x} = \frac{1}{200} \, dt
\]
\[ u = 130 - 3x \]
\[ \frac{du}{dx} = -3 \]
\[ \frac{d(3u)}{3} = dx \]

\[-\frac{1}{3} \int \frac{du}{u} = \frac{1}{200} \int dt \]

\[-\frac{1}{3} \ln |u| = \frac{t}{200} + C \]

\[-\frac{1}{3} \ln |130 - 3x| = \frac{t}{200} + C \]

\[ \text{Find } C: \text{ we know } x(0) = 0 \]

\[-\frac{1}{3} \ln |130 - 3(0)| = \frac{0}{200} + C \]

\[ C = -\frac{1}{3} \ln 130 \]

\[-\frac{1}{3} \ln |130 - 3x| = \frac{t}{200} - \frac{1}{3} \ln 130 \]

\[ \ln |130 - 3x| = -\frac{3t}{200} + \ln 130 \]

\[ |130 - 3x| = e^{-\frac{3t}{200}} + \ln 130 \]

\[ 130 - 3x = \pm e^{\ln 130} e^{-\frac{3t}{200}} \]

\[ -3x = \pm 130 e^{-\frac{3t}{200}} - 130 \]

We use \( x(0) = 0 \) to pick the negative root.

\[ x = \pm \frac{130}{3} e^{-\frac{3t}{200}} + \frac{130}{3} \]

\[ x = -\frac{130}{3} e^{-\frac{3t}{200}} + \frac{130}{3} \]

b) After 1 hour

\[ x(60) = -\frac{130}{3} e^{-\frac{3(60)}{200}} + \frac{130}{3} = 25.7 \text{ kg} \]
Direction Fields (a.k.a. Slope Fields)

Direction fields allow us to roughly plot the solution of a differential equation without actually finding the solution.

**Example:** Sketch a direction field for \( y' = x^2 + y \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>2</th>
<th>-1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

\( y' = x^2 + y \)

Sketch the curve that satisfies \( y(1) = 1 \).

**HW:** 9.3 finish it
9.4
7.7